13

The Assignment Problem

LEARNING OBJECTIVES :

After studying this unit, you will be able to :

- Use assignment technique, which is a special form of transportation problem.
- Use Hungarian Assignment Method.
- Negotiate with restrictions on assignments.
- Deal with unbalanced assignment problems.
- Prepare table with dummy rows and columns.

13.1 INTRODUCTION

The Assignement Problem is another special case of LPP. It occurs when n jobs are to be assigned to n facilities on a one-to-one basis with a view to optimising the resource required.

13.2 THE ASSIGNMENT ALGORITHM

The assingment problem can be solved by applying the following steps :

Step 1: Subtract the minimum element of each row from all the elements in that row. From each column of the matrix so obtained, subtract its minimum element. The resulting matrix is the starting matrix for the following procedure.

Step 2: Draw the minimum number of horizontal and vertical lines that cover all the zeros. If this number of lines is *n*, order of the matrix, optimal assignment can be made by skipping steps 3 and 4 and proceeding with step 5. If, however, this number is less than *n*, go to the next step.

Step 3: Here, we try to increase the number of zeros in the matrix. We select the smallest element out of these which do not lie on any line. Subtract this element from all such (uncovered) elements and add it to the elements which are placed at the intersections of the horizontal and vertical lines. Do not alter the elements through which only one line passes.

Step 4: Repeat steps 1, 2 and 3 until we get the minimum number of lines equal to *n*.

Step 5 (A) Starting with first row, examine all rows of matrix in step 2 or 4 in turn until a row containing exactly one zero is found. Surround this zero by □, indication of an assignment there. Draw a vertical line through the column containing this zero. This eliminates any confusion of making any further assignments in that column. Process all the rows in this way.

(B) Apply the same treatment to columns also. Starting with the first column, examine all columns until a column containing exactly one zero is found. Mark □ around this zero and draw a horizontal line through the row containing this marked zero. Repeat steps 5A and B, until one of the following situations arises:

(i) No unmarked (□) or uncovered (by a line) zero is left,

(ii) There may be more than one unmarked zero in one column or row. In this case, put □ around one of the unmarked zero arbitrarily and pass 2 lines in the cells of the remaining zeros in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Illustration

An Accounts Officer has 4 suboridnates and 4 tasks. The subordinates differ in efficiency. The tasks also differ in their intrinsic difficulty. His estimates of the time each would take to perform each task is given in the matrix below. How should the tasks be allocated one to one man, so that the total man hours are minimized?

	I	II	III	IV
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

Let us apply the above steps take the above example.

Solution

Step 1

By subtracting the minimum element of each row from all its elements in turn, the given matrix reduces to

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Next we subtract the minimum element of each column from all elements in turn, obtaining,

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 2

We draw the minimum number of lines to cover all zeros in the last matrix above as follows. To do so the first line is row 3 that contains the highest number of zeros. It can be seen that 4(= n) lines cover all the zeros; hence optimal assignment is possible and it is obtained by the application of step 5 straight away below.

$$
\begin{array}{cccc}
\boxed{0} & 14 & 9 & 3 \\
9 & 20 & \boxed{0} & 22 \\
23 & \boxed{0} & 3 & \cancel{0} \\
9 & 12 & 14 & \boxed{0}
\end{array}
$$

The optimal assignment, then is

$1 \rightarrow$ I, $2 \rightarrow$ III, $3 \rightarrow$ II, $4 \rightarrow$ IV.

Minimum time taken = 8 + 4 + 19 + 10 = 41 hours.

Illustration

A manager has 5 jobs to be done. The following matrix shows the time taken by the j-th job (j = 1,2...5) on the i-th machine (i = 1,2,3...5). Assign 5 jobs to the 5 machines so that the total time taken is minimized.

	Jobs 1	2	3	4	5
I	9	3	4	2	10
II	12	10	8	11	9
III	11	2	9	0	8
IV	8	0	10	2	1
V	7	5	6	2	9

Solution

Subtracting the minimum element of each row from all its elements in turn, the given matrix reduces to

$$
\begin{array}{ccccc}
7 & 1 & 2 & 0 & 8 \\
4 & 2 & 0 & 3 & 1 \\
11 & 2 & 9 & 0 & 8 \\
8 & 0 & 10 & 2 & 1 \\
5 & 3 & 4 & 0 & 7
\end{array}
$$

Now subtracting the minimum element of each column from its column, the matrix reduces to

$$
\begin{array}{ccccc}
3 & 1 & 2 & 0 & 7 \\
0 & 2 & 0 & 3 & 0 \\
7 & 2 & 9 & 0 & 7 \\
4 & 0 & 10 & 2 & 0 \\
1 & 3 & 4 & 0 & 6
\end{array}
$$

Next, we draw the minimum number of lines as follows:

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	2	0
1	3	4	0	6

Matrix A

Since there are only 3 lines (less than 5) optimal assignment cannot be made as yet. We, therefore, perform step 3 on the last matrix above. The minimum uncovered element is 1. It is subtracted from all the uncovered elements and adding it to those at intersection of two lines, giving thereby the following matrix.

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	3	0
0	2	3	0	5

As can be seen below, minimum number of lines that cover all zeros is 5.

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	3	0
0	2	3	0	5

Optimal assignment is, therefore, possible and is made as per step 5 below:

2	$\boxed{0}$	1	0	6
0	2	$\boxed{0}$	4	0
6	1	8	$\boxed{0}$	6
4	0	10	3	$\boxed{0}$
$\boxed{0}$	2	3	0	5

Optimal assignment, then is

I $\rightarrow$ 2, II $\rightarrow$ 3, III $\rightarrow$ 4, IV $\rightarrow$ 5, V $\rightarrow$ 1.

Minimum time = 3 + 8 + 0 + 1 + 7 = 19.

Illustration

5 salesmen are to be assigned to 5 districts. Estimates of sales revenue in thousands of rupees for each salesmen are given below.

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximizes the sales revenue.

Solution

In order to convert this maximization problem into a minimization problem to be able to apply the assignment algorithm, we subtract each element from the highest, 41 and obtain the following loss matrix.

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Applying step 1 to the loss matrix we derive the following matrix, in which 4 lines are drawn to cover all zeros.

8	0	0	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	0	5
11	5	0	0	1

The minimum uncovered element is 4 that is subtracted from all elements and added to all elements at intersections. This yields the following matrix in which 5 lines are needed to cover all zeros.

12	0	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Step 5 (A) and (B) is applied below to obtain the optimal assignment.

12	$\boxed{0}$	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Condition (ii) of Step B arises above; therefore cell (2,1) is arbitrarily chosen and □ put around it and a line is also drawn in the second row.

12	$\boxed{0}$	0	7	0
$\boxed{0}$	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

This process is repeated below by putting □ around cell (3,4) arbitrarily chosen. Therefore, 2 lines are drawn to cover 4th row and 3rd column.

12	$\boxed{0}$	0	7	0
$\boxed{0}$	10	8	10	0
0	8	4	2	0
23	1	$\boxed{0}$	0	5
15	5	0	0	1

The same process is continued to yield the optimal pattern as shown below:

12	$\boxed{0}$	0	7	0
$\boxed{0}$	10	8	10	0
0	8	4	2	$\boxed{0}$
23	1	$\boxed{0}$	0	5
15	5	0	$\boxed{0}$	1

Optimal assignment, then is

1 $\rightarrow$ B, 2 $\rightarrow$ A, 3 $\rightarrow$ E, 4 $\rightarrow$ C, and 5 $\rightarrow$ D.

The maximum assignment profit is given by Z = 38 + 40 + 37 + 41 + 35 = 191 thousand rupees.

13.2.1 Rationale of the Assignment Algorithm

Step 1: The relative cost of assigning facility i to job j is not changed by the subtraction of a constant from either a column or a row of the original cost matrix.

Step 2: An optimal assignment exists if total reduced cost of the assignment is zero. This is the case when the minimum number of lines necessary to cover all zeros is equal to the order of the matrix. If, however, it is less than *n,* a further reduction of the cost matrix has to be undertaken.

Step 3: The underlying logic can be explained by means of Matrix A of Example 2, in which only 3(= *n*–2) lines can be drawn.

An optimal assignment is not possible. Further reduction is necessary. This reduction is made by subtracting the smallest non-zero element from all elements in the matrix which is 1.

This yields the following matrix:

2	0	1	–1	6
–1	1	–1	2	–1
6	1	8	–1	6
3	–1	9	1	–1
0	2	3	–1	5

This matrix contains -ve values. Since the objective is to obtain an assignment with the reduced costs of zero, the -ve numbers must be eliminated. This can be done by adding 1 to each of the rows and columns crossed by 3 lines shown in matrix A of Example 2. Doing so in the above table yields the following matrix:

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	2	0
0	2	3	0	5

All this, in fact, amounts to step 3 i.e., add the least non-zero uncovered element to elements at intersections, subtract it from all the uncovered elements and leave other elements unaltered.

13.3 UNBALANCED ASSIGNMENT PROBLEMS

Like the unbalanced transportation problems there could arise unbalanced assignment problems too. They are to be handled exactly in the same manner i.e., by introducing dummy jobs or dummy men, etc. The following unbalanced problem serves as an example.

Illustration

To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars on four contemporary topics– leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be held once a week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

	Leasing	*Portfolio Management*	*Private Mutual Funds*	*Swaps and Options*
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar.

Solution

This is an unbalanced minimization assignment problem. We first of all balance it by adding a dummy topic:

	Leasing	*Portfolio Management*	*Private MutualFunds*	*Swaps and Options*	*Dummy*
Monday	50	40	60	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Subtracting the minimum element of each column from all the elements of that column, we get the following matrix :

	Leasing	*Portfolio Management*	*Private Mutual Funds*	*Swaps and Options*	*Dummy*
Monday	40	20	50	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

The minimum number of lines to cover all zeros is 4 which is less than the order of the square matrix, (i.e. 5), the above matrix will not give the optimal solution. Subtract the minimum uncovered element (=10) from all uncovered elements and add it to the elements lying on the intersection of two lines, we get the following matrix:

	Leasing	*Portfolio Management*	*Private Mutual Funds*	*Swaps and Options*	*Dummy*
Monday	30	20	40	0	0
Tuesday	20	10	20	10	0
Wednesday	40	0	10	0	0
Thursday	10	10	0	10	0
Friday	0	10	0	20	10

Since the minimum number of lines to cover all zeros is 5 which is equal to the order of the matrix, the above matrix will give the optimal solution which is given below:

	Leasing	*Portfolio Management*	*Private Mutual Funds*	*Swaps and Options*	*Dummy*
Monday	30	20	40	$\boxed{0}$	0
Tuesday	20	10	20	10	$\boxed{0}$
Wednesday	40	$\boxed{0}$	10	0	0
Thursday	10	10	$\boxed{0}$	10	0
Friday	$\boxed{0}$	10	0	20	10

and the optimal schedule is

		No. of students missing
Monday:	Swaps and options	20
Tuesday:	No seminar	0
Wednesday:	Portfolio Management	20
Thursday:	Pvt. Mutual funds	20
Friday:	Leasing	10
		70

Thus, the total number of students who will be missing at least one seminar = 70

Illustration

A solicitor's firm employs typists on hourly piece-rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to an earlier understanding only one job is given to one typist and the typist is paid for full hours even if he works for a fraction of an hour. Find the least cost allocation for the following data:

Typist	Rate per hour (Rs.)	No. of pages typed/hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of pages
P	199
Q	175
R	145
S	198
T	178

Solution

The following matrix gives the cost incurred if the *ith* typist (*i* = A, B, C, D, E) executes the *jth* job (*j* = P, Q, R, S, T):

	Job				
Typist	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Subtracting the minimum element of each row from all its elements in turn, the above matrix reduces to

	Job				
Typist	P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Now subtract the minimum element of each column from all its elements in turn, the above matrix reduces to

2	2	0	4	0
6	4	0	10	2
0	1	0	1	2
2	4	0	4	2
2	0	0	0	2

Since there are only 4 lines (< 5) to cover all zeros, optimal assignment cannot be made. The minimum uncovered element is 2.

We subtract the value 2 from all uncovered elements, add this value to all junction values and leave the other elements undisturbed. The revised matrix looks as:

2	2	2	4	0
4	2	0	8	0
0	1	2	1	2
0	2	0	2	0
2	0	2	0	2

Since the minimum number of lines required to cover all the zeros is only 4 (< 5), optimal assignment cannot be made at this stage also.

The minimum uncovered element is 1. Repeating the usual process again, we get the following matrix:

2	1	2	3	0
4	1	0	7	0
0	0	2	0	2
0	1	0	1	0
3	0	3	0	3

Since the minimum number of lines to cover all zeros is equal to 5, this matrix will give optimal solution. The optimal assignment is made in the matrix below:

Typist/Job	P	Q	R	S	T
A	2	1	2	3	$\boxed{0}$
B	4	1	0	7	0
C	8	$\boxed{0}$	2	0	2
D	$\boxed{0}$	1	0	1	0
E	3	0	3	$\boxed{0}$	3

		Cost (Rs.)
Thus typist A is given job T	:	75
Typist B is given job R	:	66
Typist C is given job Q	:	66
Typist D is given job P	:	80
Typist E is given job S	:	112
	Total Rs.	399

Note: In this case the above solution is not unique. Alternative solution also exists.

Illustration

WELLDONE Company has taken the third floor of a multi-storeyed building for rent with a view to locate one of their zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have windows, some are closer to the washrooms or to the canteen or secretarial pool. The rooms are of all different sizes and shapes. Each of the five managers was asked to rank their room preferences amongst the rooms 301, 302, 303,304 and 305. Their preferences were recorded in a table as indicated below:

MANAGER

M_1	M_2	M_3	M_4	M_5
302	302	303	302	301
303	304	301	305	302
304	305	304	304	304
	301	305	303	
		302		

Most of the managers did not list all the five rooms since they were not satisfied with some of these rooms and they have left off these from the list. Assuming that their preferences can be quantified by numbers, find out as to which manager should be assigned to which room so that their total preference ranking is a minimum.

Solution

Let us first formulate the preference ranking assignment problem.

MANAGERS

Room No.	M_1	M_2	M_3	M_4	M_5
301	–	4	2	–	1
302	1	1	5	1	2
303	2	–	1	4	–
304	~~3~~	~~2~~	~~3~~	~~3~~	~~3~~
305	–	3	4	2	–

We have to find an assignment so that total preference ranking is minimum. In a cell (-) indicates that no assignment is to be made in that particular cell. Let us assign a very large ranking value M to all such cells.

Step 1 : From each row, subtract the minimum element of that row, from all the elements of that row to get the following matrix.

MANAGERS

Room No	M_1	M_2	M_3	M_4	M_5
301	M	3	1	M	0
302	0	0	4	0	1
303	1	M	0	3	M
304	1	0	1	1	1
305	M	1	2	0	M

Draw the minimum number of lines in the above table to cover all zeros. In this case the number of such lines is five, so the above matrix will give the optimal solution. The assignment is made as below:

	MANAGERS				
Rooms No.	M_1	M_2	M_3	M_4	M_5
301	M	3	1	M	$\boxed{0}$
302	$\boxed{0}$	0	4	0	1
303	1	M	$\boxed{0}$	3	M
304	1	$\boxed{0}$	1	1	I
305	M	1	2	$\boxed{0}$	M

Thus, the assignment is

$M_1 \rightarrow 302,\ M_2 \rightarrow 304,\ M_3 \rightarrow 303,\ M_4 \rightarrow 305,\ M_5 \rightarrow 301$

and the total minimum ranking = 1 + 2 + 1 + 2 + 1 = 7

Illustration

XYZ airline operating 7 days a week has given the following timetable. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover.

Chennai-Mumbai			*Mumbai-Chennai*		
Flight Number	*Depart.*	*Arrive*	*Flight Number*	*Depart.*	*Arrive*
A1	*6 AM*	*8 AM*	*B1*	*8 AM*	*10 AM*
A2	*8 AM*	*10 AM*	*B2*	*9 AM*	*11 AM*
A3	*2 PM*	*4 PM*	*B3*	*2 PM*	*4 PM*
A4	*8 PM*	*10 PM*	*B4*	*7 PM*	*9 PM*

Solution

To begin with, let us first assume that the crew is based at Chennai. The flight A_1, which starts from Chennai at 6 AM, reaches Mumbai at 8 AM. The schedule time for the flight at Mumbai is 8 AM. Since the minimum layover time for crew is 5 hours, this flight can depart only on the next day i.e. the layover time will be 24 hours. Similarly, layover times for other flights are also calculated and given in the following table.

Crew based at Chennai				
Flight No.	B_1	B_2	B_3	B_4
A_1	24	25	6	11
A_2	22	23	28	9
A_3	16	17	22	27
A_4	10	11	16	21

The layover times for various flight connections when crew is assumed to be based at Mumbai are similarly calculated in the following table.

Crew based at Mumbai

Flight No.	B_1	B_2	B_3	B_4
A_1	20	19	14	9
A_2	22	21	16	11
A_3	28	27	22	17
A_4	10	9	28	23

Now since the crew can be based at either of the places, minimum layover times can be obtained for different flight numbers by selecting the corresponding lower value out of the above two tables. The resulting table is as given below:

	Flight No.			
Flight No.	B_1	B_2	B_3	B_4
A_1	20*	19*	6	9*
A_2	22	21*	16*	9
A_3	16	17	22	17*
A_4	10	9*	16	21

A * with an entry in the above table indicates that it corresponds to layover time when the crew is based at Mumbai. We will now apply the assignment algorithm to find the optimal solution. Subtracting the minimum element of each row from all the elements of that row, we get the following matrix.

	Flight No.			
Flight No.	B_1	B_2	B_3	B_4
A_1	~~14~~	~~13~~	~~0~~	~~3~~
A_2	13	12	7	0
A_3	~~0~~	~~1~~	~~6~~	~~1~~
A_4	~~1~~	~~0~~	~~7~~	~~12~~

Since there is a zero in each column, there is no need to perform column reduction. The minimum number of lines to cover all zeros is four which is equal to the order of the matrix. Hence, the above table will give the optimal solution. The assignment is made below:

	Flight No.			
Flight No.	B_1	B_2	B_3	B_4
A_1	14	13	$\boxed{0}$	3
A_2	13	12	7	$\boxed{0}$
A_3	$\boxed{0}$	1	6	1
A_4	1	$\boxed{0}$	7	12

The optimal assignment is

From flight No.	To flight No.	Layover time
A_1	B_3	6
A_2	B_4	9
A_3	B_1	16
A_4	B_2*	9
		40 hours

Illustration

An organisation producing 4 different products viz. A,B, C and D having 4 operators viz. P,Q,R and S, who are capable of producing any of the four products, works effectively 7 hours a day. The time (in minutes) required for each operator for producing each of the product are given in the cells of the following matrix along with profit (Rs. per unit).

Operator	Product			
	A	B	C	D
P	6	10	14	12
Q	7	5	3	4
R	6	7	10	10
S	20	10	15	15
Profit (Rs./Unit)	3	2	4	1

Find out the assignment of operators to products which will maximize the profit.

Solution

Using the information that the factory works effectively 7 hours (= 420 minutes) a day, and the time required by each operator for producing each of the products, we obtain the following production and profit matrices.

Production Matrix (units)				
Operator	Product			
	A	B	C	D
P	70	42	30	35
Q	60	84	140	105
R	70	60	42	42
S	21	42	28	28

Profit Matrix (in Rs.)				
Operator	Product			
	A	B	C	D
P	210	84	120	35
Q	180	168	560	105
R	210	120	168	42
S	63	84	112	28

In order to apply assignment algorithm for minimizing losses, let us first convert this profit matrix to a loss matrix by subtracting all the elements of the given matrix from its highest element which is equal to Rs. 560. The matrix so obtained is given below:

Operator	Product			
	A	B	C	D
P	350	476	440	525
Q	380	392	0	455
R	350	440	392	518
S	497	476	448	532

Now apply the assignment algorithm to the above loss matrix. Subtracting the minimum element of each row from all elements of that row, we get the following matrix:

Operator	Product			
	A	B	C	D
P	0	126	90	175
Q	380	392	0	455
R	0	90	42	168
S	49	28	0	84

Now subtract the minimum element of each column from all the elements of that column to get the following matrix:

Operator	Product			
	A	B	C	D
P	0	98	90	91
Q	380	364	0	371
R	0	62	42	84
S	49	0	0	0

Draw the minimum number of lines to cover all zeros. The minimum number of lines to cover all zeros is three which is less than the order of the square matrix (i.e. 4), thus the above matrix will not give the optimal solution. Subtract the minimum uncovered element (= 62) from all uncovered elements and add it to the elements lying on the intersection of two lines, we get the following matrix:

Operator	Product			
	A	B	C	D
P	0	36	90	29
Q	380	302	0	309
R	0	0	42	22
S	111	0	62	0

The minimum number of lines which cover all zeros is 4 which is equal to the order of the matrix, hence, the above matrix will give the optimal solution. Specific assignments in this case are as shown below:

Operator	Product			
	A	B	C	D
P	$\boxed{0}$	36	90	29
Q	380	302	$\boxed{0}$	309
R	0	$\boxed{0}$	42	22
S	111	0	62	$\boxed{0}$

Operator	Product	Profit (Rs.)
P	A	210
Q	C	560
R	B	120
S	D	28
Total	Profit (Rs.)	918

Illustration

A firm produces four products. There are four operators who are capable of producing any of these four products. The processing time varies from operator to operator. The firm records 8 hours a day and allow 30 minutes for lunch. The processing time in minutes and the profit for each of the products are given below:

Operators	Products			
	A	B	C	D
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs.) per unit	8	6	5	4

Find the optimal assignment of products to operators.

Solution

The firm records 8 hours a day and allows 30 minutes for lunch, hence the net working time available per day is 7 hours and 30 minutes i.e. 450 minutes. The number of units of each product which could be produced in 450 minutes by the four operators is calculated in the table given below:

Operators	Products			
	A	B	C	D
1	30	50	45	75
2	45	75	50	75
3	18	30	30	50
4	30	50	45	45
Profit (Rs.) per unit	8	6	5	4

Since we are given the profit per unit of each product, the profit matrix is computed as given below:

Operators	Profits Matrix in Rs. of Products			
	A	B	C	D
1	240	300	225	300
2	360	450	250	300
3	144	180	150	200
4	240	300	225	180

The above profit matrix is converted into a loss matrix by subtracting all the elements of the profit matrix from its highest pay off Rs. 450. The loss matrix so obtained is given below:

Operators	Loss matrix – Products			
	A	B	C	D
1	210	150	225	150
2	90	0	200	150
3	306	270	300	250
4	210	150	225	270

Let us now apply the assignment algorithm that is 'Hungarian Rule' to the above loss matrix. Accordingly, subtract the minimum element of each row from all its elements in turn, the above matrix thus reduces to

Operators	Loss matrix – Products			
	A	B	C	D
1	60	0	75	0
2	90	0	200	150
3	56	20	50	0
4	60	0	75	120

Subtract the minimum element of each column from all the elements of the column in turn. Draw the minimum number of lines in the resultant matrix so as to cover all zeros, we get

Operators	Loss matrix - Products			
	A	B	C	D
1	4	0	25	0
2	34	0	150	150
3	0	20	0	0
4	4	0	25	120

Since the minimum number of lines to cover all zeros is three which is one less than the order of the matrix, we subtract the minimum uncovered element (= 4) from all uncovered elements and add it to the elements lying at the intersection of two lines. The matrix so obtained is given below:

Operators	Loss matrix - Products			
	A	B	C	D
1	0	0	21	0
2	30	0	146	150
3	0	24	0	4
4	0	0	21	120

Since the minimum number of lines to cover all zeros is 4 which is equal to the order of the matrix, the above matrix will give optimal solution. The optimal assignments are given below:

Operators	Loss matrix - Products			
	A	B	C	D
1	$\cancel{0}$	$\cancel{0}$	21	$\boxed{0}$
2	30	$\boxed{0}$	146	150
3	$\cancel{0}$	24	$\boxed{0}$	4
4	$\boxed{0}$	$\cancel{0}$	21	120

The optimal assignment is as shown below:

Operators	Products	Profit (Rs.)
1	D	300
2	B	450
3	C	150
4	A	240
		Rs. 1140

Illustration

A manufacturing company, four has zones A, B, C, D and four sales engineers P, Q, R, S respectively for assignment. Since the zones are not equally rich in sales potential, therefore it is estimated that a particular engineer operating in a particular zone will bring the following sales:

Zone A	:	4,20,000
Zone B	:	3,36,000
Zone C	:	2,94,000
Zone D	:	4,62,000

The engineers are having different sales ability. Working under the same conditions, their yearly sales are proportional to 14, 9, 11 and 8 respectively. The criteria of maximum expected total sales is to be met by assigning the best engineer to the richest zone, the next best to the second richest zone and so on.

Find the optimum assignment and the maximum sales.

Solution

It is given that the yearly sales of four sales engineers are proportional to 14, 9, 11 and 8 respectively. The sum of proportions is 14+9+11+8=42.

Let us assume that Rs. 1,000 is equivalent to one unit. The sales units of four engineers in four different zones have been calculated as in the following table :

Sales	Zones			
Engineer	A	B	C	D
P	(14/42)×420 = 140	(14/42)×336 = 112	(14/42)×294 = 98	(14/42)×462 = 154
Q	(9/42)×420 = 90	(9/42)×336 = 72	(9/42)×294 = 63	(9/42)×462 = 99
R	(11/42)×420 = 110	(11/42)×336 = 88	(11/42)×294 = 77	(11/42)×462 = 121
S	(8/42)×420 = 80	(8/42)×336 = 64	(8/42)×294 = 56	(8/42)×462 = 88

The problem here is to find the optimum assigment in the following sales table so as to maximise the total sales of the company.

Sales engineer	Zones (Loss in thousands of rupees) A	B	C	D
P	140	112	98	154
Q	90	72	63	99
R	110	88	77	121
S	80	64	56	88

In order to apply the assignment algorithm, we will first convert this maximisation problem into a minimisation problem by subtracting all elements of the above matrix from the highest element i.e. 154. The resultant loss matrix is given below:-

Sales Engineers	Zones(Loss in thousands of rupees) A	B	C	D
P	14	42	56	0
Q	64	82	91	55
R	44	66	77	33
S	74	90	98	66

Now perform the row operations with each of the rows i.e. from all the elements of a row, subtract the minimum element of that row. The reduced matrix is a given below:

Sales Engineers	Zones(Loss in thousand's of rupees) A	B	C	D
P	14	42	56	0
Q	9	27	36	0
R	11	33	44	0
S	8	24	32	0

Now, from all the elements of a column, subtract the minimum element of that column. Repeat this operation with all the columns to get the following table:

Sales Engineers	Zones(Loss in thousands of rupees) A	B	C	D
P	6	18	24	0
Q	1	3	4	0
R	3	9	12	0
S	0	0	0	0

The minimum number of lines drawn to cover all zeros is 2 which is less than the order of the matrix (i.e.4), hence we can not make assignments. Subtract the minimum uncovered element from all the uncovered elements and add it to the elements lying at the intersection of two lines, we get:

Sales Engineers	Zones (Loss in thousand of rupees) A	B	C	D
P	5	17	23	0
Q	0	2	3	0
R	2	8	11	0
S	0	0	0	0

The minimum number of lines drawn again to cover all the zeros is 3 which is one less than the order of the matrix. Repeat the above step which gives the following table:

Sales Engineers	Zones (Losss in thousands of rupees) A	B	C	D
P	5	15	21	0
Q	0	0	1	0
R	2	6	9	0
S	2	0	0	3

The minimum number of lines to cover all zeros is still one less than the order of the matrix. Repeat the above step again, which gives the following table:

Sales Engineers	Zones (Loss in thousands of rupees) A	B	C	D
P	3	13	19	0
Q	0	0	1	2
R	0	4	7	0
S	2	0	0	5

The minimum number of lines drawn to cover all the zeros is 4 which is equal to the order of the matrix. Hence, the above table will give the optimum assignment. The assignments are as follows:

Sales Engineers	Zones (Loss in thousands of rupees A	B	C	D
P	~~8~~	$\boxed{13}$	19	$\boxed{0}$
Q	~~0~~	0	1	2
R	$\boxed{0}$	~~4~~	$\boxed{7}$	~~0~~
S	2	0	0	5

Engineers	Zones	Sales (in Rs.)
P	D	1,54,000
Q	B	72,000
R	A	1,10,000
S	C	56,000
		3,92,000

It can be seen from the above assignments that the best engineer P is assigned to the richest zone D, the next best engineer R is assigned to second richest zone A, the next best engineer Q is assigned to zone B and so on. Hence, the optimum assignment matches the company's criteria of achieving the maximum expected total sales.

SELF EXAMINATION QUESTIONS

1. Solve the following assignment problems:

(a)

	Jobs				
Men	1	2	3	4	5
1	2	9	3	7	1
2	6	8	7	6	1
3	4	6	5	3	1
4	4	2	7	3	1
5	5	3	9	5	1

(b)

8	4	2	7	1
0	9	5	5	4
3	8	9	2	6
4	3	1	0	3
9	5	8	9	5

(c)

	Jobs					
Men	1	2	3	4	5	6
1	5	0	6	8	7	4
2	5	2	3	0	6	7
3	3	4	4	3	5	2
4	3	9	7	2	7	6
5	9	8	7	8	4	5
6	1	8	7	4	2	3

2. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each machine is given in the following table:

Jobs	*Machine*			
	W	X	Y	Z
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

What are the job assignments which will minimize the cost ?

(Answer: A/W, B/X, C/Y)

3. A management consulting firm has a backlog of 4 contracts. Work on these contracts must be started immediately. Three project leaders are available for assignment to the contracts. Because of the varying work experience of the project leaders, the profit to consulting firm will vary based on the assignment as shown below. The unassigned contract can be completed by subcontracting the work to an outside consultant. The profit on the subcontract is zero.

Project	Contract			
Leader	1	2	3	4
A	13	10	9	11
B	15	17	13	20
C	6	8	11	7
Dummy	0	0	0	0

Find the optimal assignment. Note that the problem is basically not only unbalanced (though now balanced by inclusion of dummy) but also a maximization one.

Answer: A $\rightarrow$ 1, B $\rightarrow$ 4, C $\rightarrow$ 3, Dummy $\rightarrow$ 2